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Non-Fourier effects on transient temperature resulting from periodic on–off heat flux

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Abstract—The transient temperatures resulting from a periodic on–off heat flux boundary condition have many applications, including, among others, the sintering of catalysts frequently found during coke burn-off, and the use of laser pulses for annealing of semiconductors. In such situations, the duration of the pulses is so small (i.e. picosecond–nanosecond) that the classical heat diffusion phenomenon breaks down and the wave nature of energy propagation characterized by the hyperbolic heat conduction equation governs the temperature distribution in the medium. In this work, an explicit analytic solution is presented for a linear transient heat conduction problem in a semi-infinite medium subjected to a periodic on–off type heat flux at the boundary $x = 0$ by solving the hyperbolic heat conduction equation. The non-linear case allowing for the added effect of surface radiation into an external ambient is studied numerically.

INTRODUCTION

TRANSIENT heat conduction in solids subjected to a periodic on–off type surface heat flux has numerous practical applications, including the exothermic reactions present during coke burn-off, and the use of high energy pulse lasers. In the analysis of heat conduction involving extremely short times, the classical heat conduction equation breaks down. In such situations, the hyperbolic heat conduction equation, allowing for a finite speed of propagation of thermal disturbances, more accurately models the transient temperature distribution in the medium. Several authors have studied the effects of periodic on–off boundary conditions using the parabolic heat conduction equation. Carslaw and Jaeger [1] present an analytic solution for the temperature in a semi-infinite medium due to an applied heat flux that is turned off after some arbitrary time. Putterman and Guibert [2] consider a medium which is subjected to a periodically applied surface temperature and zero heat flux, and present an iterated solution for the surface temperatures, while Hein [3] has studied cases involving periodically applied surface temperatures. However, the results of these studies are not applicable to situations involving heating with extremely short pulses, such as those encountered in the exothermic reactions resulting from coke burn-off, and the pulse laser heating of semiconductors. Since the hyperbolic model includes a build-up period for the establishment of heat flow resulting from a thermal disturbance, the non-Fourier effects on temperature transients are expected to be strongly pronounced for the temperature of the surface at which on–off type pulsed heating of an extremely short pulse period is applied.

The non-Fourier effects on temperature transients resulting from a continuously applied surface heat flux as well as a single volumetric pulse heat source applied within the medium have been studied [4–7]. Chan *et al.* [8] stated that hyperbolic heat conduction may have significant effects on determining the temperature rise of crystals caused by exothermic reactions. In this work, the non-Fourier effects resulting from an on–off type periodic heating of an extremely short pulse period applied at the surface of a solid are studied by solving the hyperbolic heat conduction equation and making use of Duhamel's theorem. The existing proofs of Duhamel's theorem are all for parabolic heat conduction problems. Therefore, before applying Duhamel's theorem, it is shown that Duhamel's theorem is also applicable for the hyperbolic heat conduction equation.

When the surface temperature is sufficiently high, as may often be the case in the above-mentioned applications, the radiation losses from the surface into the external ambient become important. In such situations, the problem becomes nonlinear because of the radiation boundary condition. A numerical scheme is thus used to solve the resulting non-linear periodic problem with surface radiation.

NOMENCLATURE

c	speed of propagation of thermal front
C_p	specific heat
f_0	reference heat flux
$f(\xi)$	dimensionless applied surface heat flux, $g(t)/f_0$
ΔF_i	dimensionless step change in surface heat flux for linear problem
$g(t)$	applied surface heat flux
k	thermal conductivity
N	conduction-to-radiation parameter, $(kT_r c/\alpha)/\sigma T_r^4$
P	period of on–off surface heat flux
$q(x,t)$	heat flux
$Q(\eta,\xi)$	dimensionless heat flux, q/f_0
t	time
$T(x,t)$	temperature
T_r	reference temperature, $(f_0/\sigma)^{1/4}$
$U(z)$	unit step function
x	physical distance.

Greek symbols

α	reference thermal diffusivity
β	fraction of period P that surface heat flux is nonzero
ε_0	surface emissivity at $\eta = 0$
η	dimensionless spacial distance, $cx/2\alpha$
$\theta(\eta,\xi)$	dimensionless temperature, T/T_r
θ_0	dimensionless initial temperature
λ_i	dimensionless times at which step changes in surface heat flux occur
ξ	dimensionless time, $c^2 t/2\alpha$
ρ	density
σ	Stefan–Boltzmann constant
τ	relaxation time, α/c^2
$\Psi(\eta,\xi)$	dimensionless temperature.

ANALYSIS

Consider a semi-infinite region, initially at zero temperature, where a periodic on–off heat flux is applied at the boundary surface $x = 0$ for times $t > 0$. Assuming constant properties, the mathematical formulation of the hyperbolic heat conduction problem consists of the energy and non-Fourier heat flux equations

$$\rho C_p \frac{\partial T(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0, \quad x > 0, t > 0 \quad (1a)$$

$$\tau \frac{\partial q(x,t)}{\partial t} + q(x,t) = -k \frac{\partial T(x,t)}{\partial x}, \quad x > 0, t > 0 \quad (1b)$$

where $T(x,t)$ is the temperature, $q(x,t)$ the heat flux, $\tau = \alpha/c^2$ the relaxation time, ρ the density, and C_p the specific heat. Clearly, as $c \to \infty$, $\tau \to 0$ and the Fourier heat flux equation is obtained, thus yielding the parabolic heat conduction theory. The boundary and initial conditions are taken as

$$q(0,t) = g(t), \quad x = 0 \quad (1c)$$

$$T(x,t) = 0, \quad x \to \infty \quad (1d)$$

$$T(x,0) = 0, \quad t = 0 \quad (1e)$$

$$q(x,0) = 0, \quad t = 0. \quad (1f)$$

Equations (1a) and (1b) are combined and then nondimensionalized. We obtain

$$\frac{\partial^2\theta(\eta,\xi)}{\partial\xi^2} + 2\frac{\partial\theta(\eta,\xi)}{\partial\xi} = \frac{\partial^2\theta(\eta,\xi)}{\partial\eta^2}, \quad 0 < \eta < \infty, \xi < 0 \quad (2a)$$

$$Q(0,\xi) = F(\xi), \quad \eta = 0 \quad (2b)$$

$$\theta(\eta,\xi) = 0, \quad \eta \to \infty \quad (2c)$$

$$\theta(\eta,0) = 0, \quad \xi = 0 \quad (2d)$$

$$Q(\eta,0) = 0, \quad \xi = 0 \quad (2e)$$

where

$$F(\xi) = \begin{cases} f(\xi) & \text{for the linear problem} \\ f(\xi) - \varepsilon_0\theta^4(0,\xi) & \text{for the non-linear problem} \end{cases} \quad (3a)$$

$$f(\xi) = g(t)/f_0 \quad (3b)$$

$$\theta = T/T_r \quad (3c)$$

$$Q = q/f_0 \quad (3d)$$

$$\eta = cx/2\alpha \quad (3e)$$

$$\xi = c^2 t/2\alpha \quad (3f)$$

$$T_r = (f_0/\sigma)^{1/4} \quad (3g)$$

$$N = (T_r kc/\alpha)/\sigma T_r^4 \quad (3h)$$

f_0 = reference heat flux.

We consider below the solution of the above problem for both the linear and non-linear cases.

Linear problem

To solve the periodic problem for the linear case, we consider a function $\Psi(\eta,\xi)$ satisfying an auxiliary problem similar to the one given by equation (2), but subjected to a continuous surface heat flux boundary condition $F(\xi) = f_0$. The solution to this auxiliary problem is given by [4]

$$\Psi(\eta,\xi) = (f_0/N)\left\{e^{-\xi} I_0[\sqrt{(\xi^2-\eta^2)}] + 2\int_\eta^\xi e^{-\tau} I_0[\sqrt{(\tau^2-\eta^2)}]\,d\tau\right\} U(\xi-\eta) \quad (4a)$$

where $U(z)$ is the unit step function

$$U(z) = \begin{cases} 1, & z > 0 \\ 0, & z \leqslant 0. \end{cases} \quad (4b)$$

The solution of problem (2) can be related to the solution of the auxiliary problem given by equation (4) by Duhamel's theorem given in ref. [9] as (see appendix for the proof of the validity of Duhamel's theorem for the case of hyperbolic heat conduction)

$$\theta(\eta,\xi) = \theta_0 + \int_0^\xi F(\lambda)\frac{\partial \Psi(\eta,\xi-\lambda)}{\partial \xi}\,d\lambda \quad (5a)$$

where θ_0 is the initial condition, which is zero for the problem given by equation (2), and $F(\lambda)$ is the non-homogeneous part of the boundary condition. If the boundary condition function $F(\lambda)$ has discontinuities, equation (5a) is integrated by parts, and the resulting alternative form of Duhamel's theorem becomes

$$\theta(\eta,\xi) = \theta_0 + \int_0^\xi \Psi(\eta,\xi-\lambda)\frac{dF(\lambda)}{d\lambda}\,d\lambda + \sum_{i=0}^{N-1} \Psi(\eta,\xi-\lambda_i)\,\Delta F_i \quad (5b)$$

where the λ_is are the times at which there is a step change in the surface heat flux (i.e. turned on and off for the cases studied here), the ΔF_is are the magnitudes of the step changes in the surface heat flux, and $\Psi(\eta,\xi)$ is the fundamental solution with constant unit input. For the problem under consideration, the first term in equation (5b) does not contribute anything because the initial temperature θ_0 is zero. The second term vanishes since the derivative of the surface heat flux is zero over each interval. Therefore, only the third term contributes to the temperature distribution, and it represents the effects of the on–off step changes in the surface heat flux. Then, using the auxiliary solution $\Psi(\eta,\xi)$ given by equation (4), in Duhamel's theorem, equation (5b), we obtain the temperature distribution in a semi-infinite medium (or a finite medium before reflection) for a periodic on–off type surface heat flux in the form

$$\theta(\eta,\xi) = \sum_{i=0}^{\infty} (\Delta F_i/N) U[\xi-\lambda_i] \times \left\{e^{-(\xi-\lambda_i)} I_0\{\sqrt{[(\xi-\lambda_i)^2-\eta^2]}\} + 2\int_\eta^{(\xi-\lambda_i)} e^{-\tau} I_0[\sqrt{(\tau^2-\eta^2)}]\,d\tau\right\} U(\xi-\lambda_i-\eta). \quad (6)$$

For the periodic problem with period P where the duration of heating is equal to βP, the parameter λ_i is defined as

$$\lambda_i = \begin{cases} [i/2]P, & \text{for } i = 0,2,4,6,\ldots \\ [(i-1)/2+\beta]P, & \text{for } i = 1,3,5,7,\ldots \end{cases} \quad (7a)$$

and ΔF_i is related to β, the fraction of the period P that the surface heat flux is nonzero, by

$$\Delta F_i = \frac{(-1)^i}{\beta}. \quad (7b)$$

Equation (7b) implies that the total amount of energy supplied to the surface during each period is equal to that which would have been supplied by a constant heat flux of unity applied over the entire period, P. The term $(-1)^i$ represents the on–off aspect of the heating.

For each case considered, the hyperbolic solution is compared to the corresponding parabolic solution, which is obtained in a similar manner to the hyperbolic solution, and is given by

$$\theta(\eta,\xi) = 2\sqrt{2}\sum_{i=0}^{\infty} (\Delta F_i/N)\sqrt{(\xi-\lambda_i)} \times \operatorname{ierfc}\left(\frac{\eta}{\sqrt{[2(\xi-\lambda_i)]}}\right) U(\xi-\lambda_i) \quad (8a)$$

where

$$\operatorname{ierfc}(z) = \frac{1}{\sqrt{\pi}}e^{-z^2} - (z)\operatorname{erf}(z) \quad (8b)$$

and ΔF_i and λ_i are as defined earlier.

Non-linear problem

The problem becomes non-linear when the radiation effects are included at the boundary surface. Here we consider a situation in which a periodic on–off heat flux is applied to the boundary, while the surface dissipates heat by radiation into an ambient at zero temperature. The energy and non-Fourier flux equations (1a) and (1b) are given, respectively, in dimensionless form as

$$\frac{\partial \theta}{\partial \xi} + \frac{1}{N}\frac{\partial Q}{\partial \eta} = 0, \quad \eta > 0, \xi > 0 \quad (9a)$$

$$\frac{\partial Q}{\partial \xi} + N\frac{\partial \theta}{\partial \eta} + 2Q = 0, \quad \eta > 0, \xi > 0 \quad (9b)$$

while the radiation boundary condition for this situation is given by

$$Q(0,\xi) = f(\xi) - \varepsilon_0\theta^4(0,\xi), \quad \eta = 0. \quad (10a)$$

The applied surface heat flux $f(\xi)$ is defined as

$$f(\xi) = \begin{cases} 1/\beta, & [j-1]P < \xi \leqslant [(j-1)+\beta]P \\ 0, & [(j-1)+\beta]P < \xi \leqslant jP \end{cases} \qquad j = 1,2,3,\ldots \quad (10b)$$

with j representing the number of periods. We note that as in the case of the linear problem, the total energy supplied, $f(\xi)$, during each period is equal regardless of the value of β. In addition, ε_0 is the surface emissivity and N is the conduction-to-radiation parameter.

The above hyperbolic problem with surface radiation and periodic surface heat flux is solved numerically by MacCormack's predictor–corrector scheme with the use of the modified equation as described in ref. [10]. The scheme has been shown to handle the sharp wave fronts of hyperbolic heat conduction quite well [7]. To use MacCormack's method, equations (9a) and (9b) are first written in vector form as

$$\frac{\partial \mathbf{E}}{\partial \xi}+\frac{\partial \mathbf{F}}{\partial \eta}+\mathbf{H}=0, \quad \eta>0, \xi>0 \tag{11a}$$

where

$$\mathbf{E}=\begin{vmatrix}\theta \\ Q\end{vmatrix} \tag{11b}$$

$$\mathbf{F}=\begin{vmatrix}Q/N \\ N\theta\end{vmatrix} \tag{11c}$$

$$\mathbf{H}=\begin{vmatrix}0 \\ 2Q\end{vmatrix}. \tag{11d}$$

Then, MacCormack's method is applied to equations (11) to yield the following finite-difference formulation:

predictor

$$(\tilde{\mathbf{E}}_i)^{n+1}=(\mathbf{E}_i)^n-\frac{\Delta\xi}{\Delta\eta}(\mathbf{F}_{i+1}-\mathbf{F}_i)^n-\Delta\xi(\mathbf{H}_i)^n; \tag{12a}$$

corrector

$$(\mathbf{E}_i)^{n+1}=\frac{1}{2}\left[(\mathbf{E}_i)^n+(\tilde{\mathbf{E}}_i)^{n+1} -\frac{\Delta\xi}{\Delta\eta}(\tilde{\mathbf{F}}_i-\tilde{\mathbf{F}}_{i-1})^{n+1}-\Delta\xi(\tilde{\mathbf{H}}_i)^{n+1}\right]. \tag{12b}$$

Here, subscript i denotes the grid point in the space domain, superscript n denotes the time level, the tilde denotes the predicted value at the time level $n+1$, and $\Delta\eta$ and $\Delta\xi$ are the space and time steps, respectively. In this formulation, forward differencing is used in the predictor, while backward differencing is used in the corrector. In the present analysis, the modified equation approach (as discussed in ref. [10]) is used to reduce the magnitude of the truncated error terms. This involves subtracting the leading error terms of the modified equation from the finite-difference scheme. When this is done, equation (11) takes the form

$$\frac{\partial \mathbf{E}}{\partial \xi}+\frac{\partial \mathbf{F}_0}{\partial \eta}+\mathbf{H}=0 \tag{13a}$$

where $\mathbf{F}_0$ replaces $\mathbf{F}$, and is defined as

$$\mathbf{F}_0=\mathbf{F}-\mathbf{\Delta}_2-\mathbf{\Delta}_3 \tag{13b}$$

$$\mathbf{\Delta}_2=\frac{1}{3!}(\Delta\eta)^2\left((1-\nu^2)\frac{\partial^2\mathbf{F}}{\partial\eta^2}\right) \tag{13c}$$

$$\mathbf{\Delta}_3=\frac{1}{4!}(\Delta\eta)^3\left(3\nu(1-\nu^2)\frac{\partial^3\mathbf{E}}{\partial\eta^3}\right). \tag{13d}$$

The accuracy of the numerical scheme used here was verified for the periodic problem by comparing the numerical solution of the linear periodic problem with the analytical solution given by equation (6). In the numerical solution of the linear periodic problem, spikes appeared at the discontinuities. These spikes are in part due to the use of a low Courant number, $\nu=\Delta\xi/\Delta\eta$, needed to obtain stable solutions with the pulsed boundary condition. The low Courant number increases the magnitude of the error terms in the numerical solution. The spikes are a dispersive effect resulting from the odd derivative error terms. This dispersive effect can be reduced by increasing $\Delta\eta$ at the expense of increasing the dissipative effect of the even derivative error terms which in turn reduces the temperature gradients at the wave fronts [7]. In the present investigation, the use of smaller mesh intervals appeared to produce very good agreement between the numerical and exact analytic solutions. The numerical and analytical solutions agreed quite well everywhere except at these spikes.

The hyperbolic solution with surface radiation is compared to the corresponding parabolic solution with surface radiation. When surface radiation is included in the formulation, both the parabolic and the hyperbolic problems become nonlinear; therefore they are solved numerically. A central differenced, implicit, finite-difference scheme with iterations due to the radiation boundary condition is used to calculate the temperatures from the parabolic heat conduction equation.

RESULTS AND DISCUSSION

The surface and medium temperatures are obtained by the use of the hyperbolic and parabolic heat conduction equations for a semi-infinite medium with a periodic on–off surface heat flux. For the cases involving no surface radiation, the analytic solutions given by equations (6) and (8a) are used to obtain the temperature distributions, while the numerical solutions are used for the cases involving surface radiation. The temperature transients for the cases $\beta=0.5$ and 1.0 are compared, with $\beta=1.0$ corresponding to a constant heat flux of unity applied at the boundary over the entire period. For all cases considered, we have taken the length of the period to be $P=0.1$, and the conduction-to-radiation parameter N to be equal to unity. For each case, a heat flux of strength equal to $1/\beta$ stayed on for a time of βP from the beginning of each period P. Thus, the total heat flux supplied to the medium remained the same for each period P regardless of the value of β.

Figure 1 shows the hyperbolic and parabolic surface

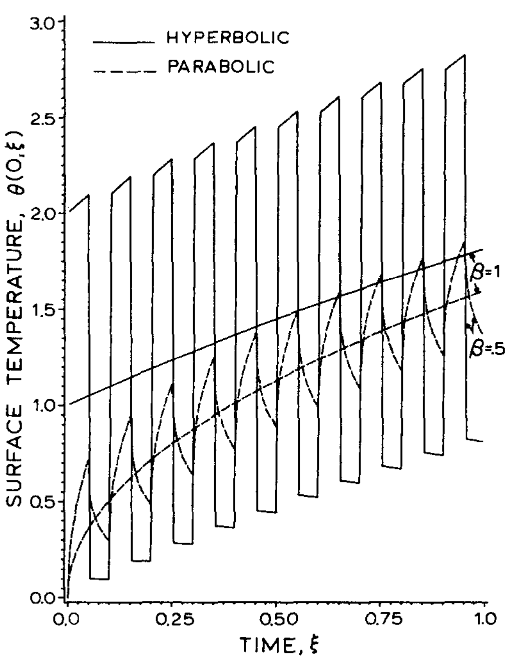

FIG. 1. Analytic comparison of the effects of periodic surface heat flux on hyperbolic and parabolic surface temperatures with no surface radiation (i.e. $\varepsilon_0 = 0$).

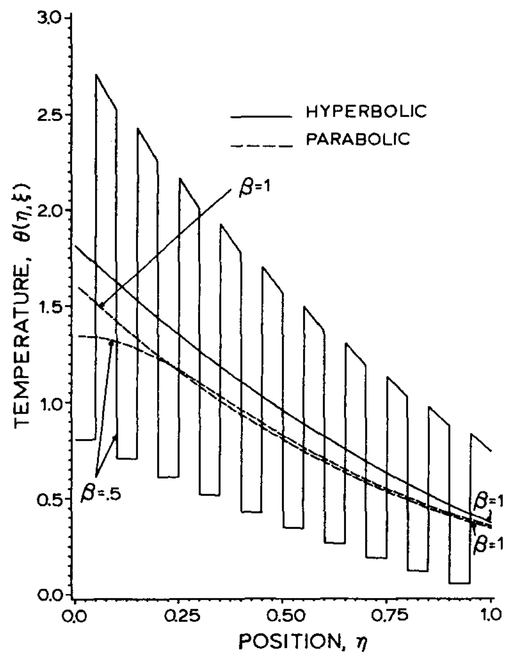

FIG. 2(a). Analytic comparison of the effects of periodic surface heat flux on hyperbolic and parabolic medium temperatures at time $\xi = 1$ with no surface radiation (i.e. $\varepsilon_0 = 0$).

temperatures for $\beta = 0.5$ and 1.0. The surface temperatures for $\beta = 0.5$ for both the parabolic and hyperbolic cases oscillate around the $\beta = 1.0$ solution. However, we notice surface temperatures are much higher for the hyperbolic solution than the parabolic solution. In both the hyperbolic and parabolic solutions, the maximum surface temperature for $\beta = 0.5$ is $1/\beta = 2.0$ times that of the constant flux ($\beta = 1.0$) solution during the time $\xi \leqslant \beta P$. The parabolic solutions for all β, however, approach zero as the time approaches zero; but this is not the case with the hyperbolic solutions, which approach a value of $1/\beta$ as the time approaches zero. Thus, with $\beta = 0.5$, the surface temperature from the hyperbolic problem at $\xi = 0^+$ is equal to 2.0. Similarly, if $\beta = 0.1$, the hyperbolic surface temperature would be equal to 10.0 at $\xi = 0^+$. As a result, at early times there is a large difference between temperatures for different values of β in the hyperbolic solution, but not in the parabolic solution (since it approaches zero). We also notice that the amplitudes of the oscillations are much larger for the hyperbolic solution than the parabolic solution. The temperature distributions were also calculated for a larger time, $\xi = 10$. The temperature amplitudes for $\beta = 0.5$ remained approximately unchanged, although the absolute temperatures continued to increase for both the hyperbolic and parabolic solutions. At $\xi = 10$, the $\beta = 1$ solutions for the surface temperature attained values about $\theta(0, \xi) = 5.0$, approximately at the midpoint of the oscillating temperatures.

Figure 2(a) shows a comparison of the medium temperatures for both the hyperbolic and parabolic problems with $\beta = 0.5$ and 1.0 at time $\xi = 1$. We notice that in the region away from the surface, the periodic surface heat flux has little effect on the parabolic temperatures. However, for the hyperbolic case, the temperature oscillations continue to propagate into the medium (though with decreasing amplitude), and as a result there is a noticeable difference between the periodic and constant flux temperature solutions. Figure 2(b) shows the behavior of the medium temperatures at time $\xi = 10.0$ with $\beta = 0.5$. It is interesting to note that the hyperbolic and parabolic temperatures have nearly converged by the position $\eta = 5$. Although only the solution for a periodic flux with $\beta = 0.5$ is shown, the temperature distributions with $\beta = 1.0$ for both the hyperbolic and parabolic cases are very similar to the parabolic case with $\beta = 0.5$. It has been shown in previous works [7, 11, 12] that the hyperbolic and parabolic solutions converge for large times for various types of problems. Figure 2(b) shows that the hyperbolic and parabolic temperature distributions with periodic surface heat flux do begin to converge for relatively large times. However, for short times, the parabolic solution underestimates the maximum temperatures by a large amount.

The effects of surface radiation on the surface temperature are shown in Fig. 3 for $\beta = 0.5$ and 1.0. Again, the hyperbolic and parabolic solutions are both plotted for comparison. Due to the inclusion of surface radiation, both hyperbolic and parabolic solutions are calculated numerically. In Fig. 3, spikes occurring at each discontinuity in the numerical solution result, as discussed previously, from the use of the low Courant number. We notice that with surface radiation, the hyperbolic and parabolic solutions converge more rapidly. This was also demonstrated for

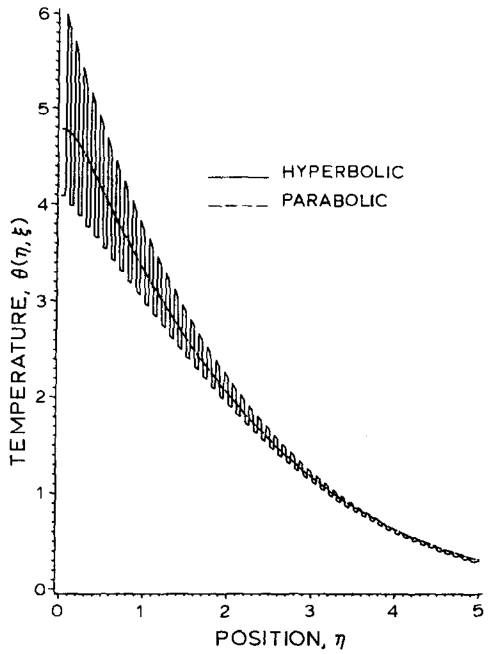

FIG. 2(b). Analytic comparison of the effects of periodic surface heat flux on hyperbolic and parabolic medium temperatures for $\beta = 0.5$ at time $\xi = 10$ with no surface radiation (i.e. $\varepsilon_0 = 0$).

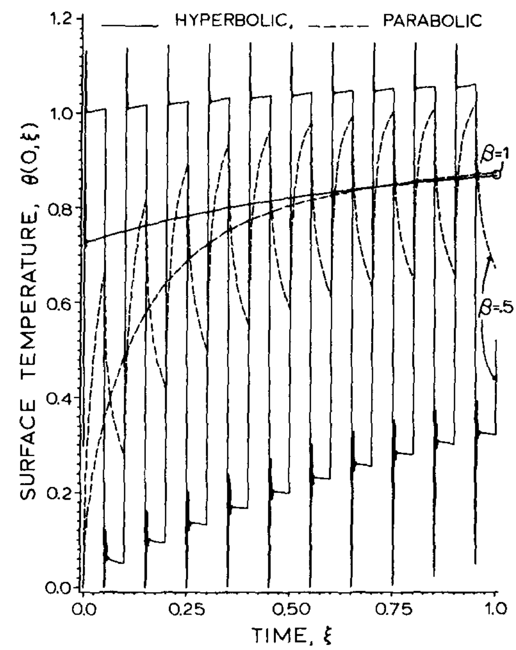

FIG. 3. Comparison of the effects of periodic surface heat flux and surface radiation on the hyperbolic and parabolic surface temperatures with $\varepsilon_0 = 1$.

several non-periodic cases in an earlier work [11]. The surface radiation not only lowers the temperatures at the surface $\eta = 0$, but also brings the maximum hyperbolic and parabolic temperatures much closer together. This is due in part to the fact that in the hyperbolic solution, more energy is radiated to the ambient than in the parabolic solution because the surface temperatures are larger. Thus, when surface radiation is present, the parabolic heat conduction equation does not underestimate the early time temperatures as much as when no surface radiation is present. The surface temperatures were also calculated to a time $\xi = 10$. At time $\xi = 10$, the hyperbolic and parabolic temperatures for $\beta = 1$ have converged, and with the surface temperature near $\theta(0, \xi) = 0.95$, seem to be nearing a steady-state value. The periodic cases also appear to have reached a quasi-equilibrium temperature distribution, with the parabolic temperature oscillating between 0.75 and 1.05, while the hyperbolic temperature oscillates between 0.55 and 1.1.

The medium temperatures at time $\xi = 1$ are shown in Fig. 4 with the presence of surface radiation for $\varepsilon_0 = 1.0$. The shape of the curves seem to resemble those in Fig. 2(a) except that the magnitudes of the temperatures are reduced as a result of the surface radiation. The time for convergence between the $\beta = 1$ solutions is also much less due to the radiation losses. The spikes on the temperature pulses are again a result of the dispersive effect of the leading error terms.

In conclusion, the parabolic heat conduction equation underestimates the temperatures near the surface for early times when a periodic surface heat flux of extremely short periods is applied. The hyperbolic and parabolic solutions converge in the interior region both with increasing distance from the surface and increasing time. Also, the convergence of the hyperbolic and parabolic solutions occurs more rapidly with increasing surface radiation.

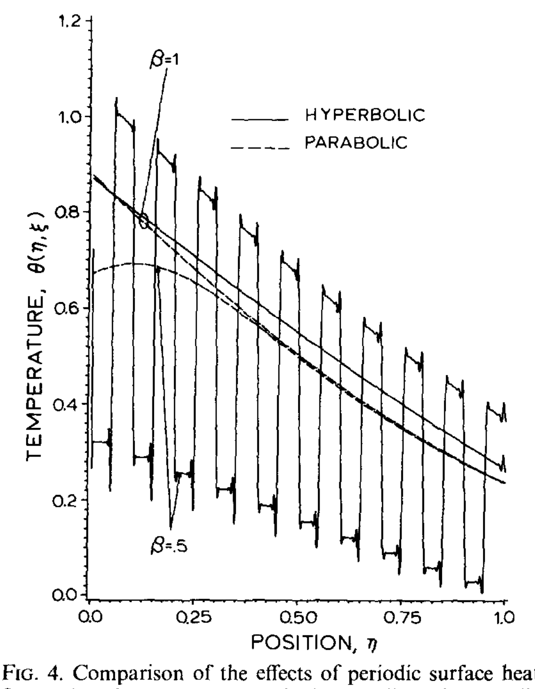

FIG. 4. Comparison of the effects of periodic surface heat flux and surface radiation on the hyperbolic and parabolic medium temperatures with $\varepsilon_0 = 1$ at time $\xi = 1$.

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APPENDIX

In the following analysis, we follow the formalism used in ref. [9] to prove Duhamel's theorem for the case of periodic hyperbolic heat conduction. The governing equations are

$$\nabla^2 T(\mathbf{r},t) + \frac{1}{k}\,g(\mathbf{r},t) = \frac{1}{\alpha}\left(\tau\frac{\partial^2 T(\mathbf{r},t)}{\partial t^2} + \frac{\partial T(\mathbf{r},t)}{\partial t}\right) \quad \text{in region } R,\ t>0 \tag{A1a}$$

$$q(\mathbf{r},t) + h_i T(\mathbf{r},t) = f_i(\mathbf{r},t) \quad \text{on boundary } S_i,\ t>0 \tag{A1b}$$

$$T(\mathbf{r},t) = F(\mathbf{r}) \quad \text{in region } R,\ t=0 \tag{A1c}$$

$$\frac{\partial T(\mathbf{r},t)}{\partial t} = 0 \quad \text{in region } R,\ t=0 \tag{A1d}$$

where $q(\mathbf{r},t)$ is obtained from

$$\tau\frac{\partial q(\mathbf{r},t)}{\partial t} + q(\mathbf{r},t) = -k\frac{\partial T(\mathbf{r},t)}{\partial n_i} \tag{A1e}$$

with

$$q(\mathbf{r},0) = q_0(\mathbf{r}) \quad \text{in region } R,\ t=0. \tag{A1f}$$

We now consider the following auxiliary problem in which $\phi(\mathbf{r},t,\gamma)$ is the solution of problem (A1) on the assumption that the non-homogeneous terms $f_i(\mathbf{r},t)$ and $g(\mathbf{r},t)$ do not depend on time

$$\nabla^2\phi(\mathbf{r},t,\gamma) + \frac{1}{k}\,g(\mathbf{r},\gamma) = \frac{1}{\alpha}\left(\tau\frac{\partial^2\phi(\mathbf{r},t,\gamma)}{\partial t^2} + \frac{\partial\phi(\mathbf{r},t,\gamma)}{\partial t}\right) \quad \text{in region } R,\ t>0 \tag{A2a}$$

$$\Upsilon(\mathbf{r},t,\gamma) + h_i\phi(\mathbf{r},t,\gamma) = f_i(\mathbf{r},\gamma) \quad \text{on boundary } S_i,\ t>0 \tag{A2b}$$

$$\phi(\mathbf{r},t,\gamma) = F(\mathbf{r}) \quad \text{in region } R,\ t=0 \tag{A2c}$$

$$\frac{\partial\phi(\mathbf{r},t,\gamma)}{\partial t} = 0 \quad \text{in region } R,\ t=0 \tag{A2d}$$

where $\Upsilon(\mathbf{r},t,\gamma)$ is obtained from

$$\tau\frac{\partial\Upsilon(\mathbf{r},t,\gamma)}{\partial t} + \Upsilon(\mathbf{r},t,\gamma) = -k\frac{\partial\phi(\mathbf{r},t,\gamma)}{\partial n_i} \tag{A2e}$$

with

$$\Upsilon(\mathbf{r},t,\gamma) = q_0(\mathbf{r}) \quad \text{in region } R,\ t=0. \tag{A2f}$$

The Laplace transform of equations (A1a)–(A1f) with respect to t gives

$$\nabla^2\bar{T}(\mathbf{r},s) + \frac{1}{k}\,\bar{g}(\mathbf{r},s) = \frac{1}{\alpha}\left(\tau s^2\bar{T}(\mathbf{r},s) - \tau sF(\mathbf{r}) - \tau\frac{\partial T(\mathbf{r},0)}{\partial t} + s\bar{T}(\mathbf{r},s) - F(\mathbf{r})\right) \quad \text{in region } R \tag{A3a}$$

$$\bar{q}(\mathbf{r},s) + h_i\bar{T}(\mathbf{r},s) = \bar{f}_i(\mathbf{r},s) \quad \text{on boundary } S_i \tag{A3b}$$

$$\tau[s\bar{q}(\mathbf{r},s) - q_0(\mathbf{r})] + \bar{q}(\mathbf{r},s) = -k\frac{\partial\bar{T}(\mathbf{r},s)}{\partial n_i} \tag{A3c}$$

where s is the Laplace transform variable. Similarly, the Laplace transform of the auxiliary problem (A2) with respect to t becomes

$$\nabla^2\bar{\phi}(\mathbf{r},s,\gamma) + \frac{1}{sk}\,g(\mathbf{r},\gamma) = \frac{1}{\alpha}\left(\tau s^2\bar{\phi}(\mathbf{r},s,\gamma) - \tau sF(\mathbf{r}) - \tau\frac{\partial\phi(\mathbf{r},0,0)}{\partial t} + s\bar{\phi}(\mathbf{r},s,\gamma) - F(\mathbf{r})\right) \quad \text{in region } R \tag{A4a}$$

$$\bar{\Upsilon}(\mathbf{r},s,\gamma) + h_i\bar{\phi}(\mathbf{r},s,\gamma) = \frac{1}{s}f_i(\mathbf{r},\gamma) \quad \text{on boundary } S_i \tag{A4b}$$

$$\tau[s\bar{\Upsilon}(\mathbf{r},s,\gamma) - q_0(\mathbf{r})] + \bar{\Upsilon}(\mathbf{r},s,\gamma) = -k\frac{\partial\bar{\phi}(\mathbf{r},s,\gamma)}{\partial n_i}. \tag{A4c}$$

We now operate on equations (A4) with the operator

$$\int_0^\infty e^{-s\gamma}\,d\gamma$$

and utilize the definition of the Laplace transform and obtain

$$\nabla^2\bar{\bar{\phi}}(\mathbf{r},s) + \frac{1}{sk}\,\bar{g}(\mathbf{r},s) = \frac{1}{\alpha}\left(\tau s^2\bar{\bar{\phi}}(\mathbf{r},s) - \tau F(\mathbf{r}) - \frac{\tau}{s}\frac{\partial\phi(\mathbf{r},0,0)}{\partial t} + s\bar{\bar{\phi}}(\mathbf{r},s) - F(\mathbf{r})/s\right) \quad \text{in region } R \tag{A5a}$$

$$\bar{\bar{\Upsilon}}(\mathbf{r},s) + h_i\bar{\bar{\phi}}(\mathbf{r},s) = \frac{1}{s}\bar{f}_i(\mathbf{r},s) \quad \text{on boundary } S_i \tag{A5b}$$

$$\tau[s\bar{\bar{\Upsilon}}(\mathbf{r},s) - q_0(\mathbf{r})/s] + \bar{\bar{\Upsilon}}(\mathbf{r},s) = -k\frac{\partial\bar{\bar{\phi}}(\mathbf{r},s)}{\partial n_i} \tag{A5c}$$

since

$$\bar{\bar{\phi}}(\mathbf{r},s,s) = \bar{\bar{\phi}}(\mathbf{r},s). \tag{A5d}$$

Also, $\bar{\bar{\phi}}(\mathbf{r},s)$ is defined as

$$\bar{\bar{\phi}}(\mathbf{r},s) = \int_0^\infty\int_0^\infty e^{-(\gamma+t)s}\,\phi(\mathbf{r},t,\gamma)\,dt\,d\gamma. \tag{A6}$$

Now multiply both sides of equations (A5a)–(A5c) by s

$$\nabla^2 s\bar{\bar{\phi}}(\mathbf{r},s) + \frac{1}{k}\bar{g}(\mathbf{r},s) = \frac{1}{\alpha}\bigg(\tau s^2[s\bar{\bar{\phi}}(\mathbf{r},s)] - \tau sF(\mathbf{r}) - \tau\frac{\partial\phi(\mathbf{r},0)}{\partial t} + s[s\bar{\bar{\phi}}(\mathbf{r},s)] - F(\mathbf{r})\bigg)$$

$$\text{in region } R \quad \text{(A7a)}$$

$$s\bar{\bar{\Upsilon}}(\mathbf{r},s) + h_i s\bar{\bar{\phi}}(\mathbf{r},s) = \bar{\bar{f}}_i(\mathbf{r},s) \quad \text{on boundary } S_i \quad \text{(A7b)}$$

$$\tau[s^2\bar{\bar{\Upsilon}}(\mathbf{r},s) - q_0(\mathbf{r})] + s\bar{\bar{\Upsilon}}(\mathbf{r},s) = -k\frac{\partial s\bar{\bar{\phi}}(\mathbf{r},s)}{\partial n_i}. \quad \text{(A7c)}$$

A comparison of equations (A3a)–(A3c) and (A7a)–(A7c) reveals that they are identical problems if

$$\bar{\bar{\Upsilon}}(\mathbf{r},s) = s\bar{\bar{\phi}}(\mathbf{r},s) \quad \text{(A8a)}$$

yielding

$$\bar{\bar{\Upsilon}}(\mathbf{r},s) = s\int_0^\infty\int_0^\infty e^{-(\gamma+t)s}\phi(\mathbf{r},t,\gamma)\,dt\,d\gamma \quad \text{(A8b)}$$

and

$$\bar{q}(\mathbf{r},s) = s\bar{\bar{\Upsilon}}(\mathbf{r},s) \quad \text{(A8c)}$$

yielding

$$\bar{q}(\mathbf{r},s) = s\int_0^\infty\int_0^\infty e^{-(\gamma+t)s}\bar{\bar{\Upsilon}}(\mathbf{r},t,\gamma)\,dt\,d\gamma. \quad \text{(A8d)}$$

Now consider a general function $\phi(\mathbf{r},t,\gamma)$ where the convolution is

$$\phi^*(\mathbf{r},t) = \int_0^t \phi(\mathbf{r},t-\gamma,\gamma)\,d\gamma \quad \text{(A9)}$$

and the Laplace transform of this generalized convolution becomes

$$\mathscr{L}[\phi^*(\mathbf{r},t)] = \bar{\phi}^*(\mathbf{r},s) \quad \text{(A10)}$$

$$= \int_0^\infty\int_0^\infty e^{-(\gamma+t)s}\phi(\mathbf{r},t,\gamma)\,dt\,d\gamma.$$

A comparison of equations (A8a), (A8b) and (A10) yields

$$\mathscr{L}[T(\mathbf{r},t)] = s\bar{\phi}^*(\mathbf{r},s). \quad \text{(A11)}$$

By the definition of the Laplace transform, we have

$$\mathscr{L}[T(\mathbf{r},t)] = \mathscr{L}\left(\frac{\partial\phi^*(\mathbf{r},t)}{\partial t}\right) \quad \text{(A12)}$$

since

$$\phi^*(\mathbf{r},0) = 0. \quad \text{(A13)}$$

Now invert equation (A12)

$$T(\mathbf{r},t) = \frac{\partial\phi^*(\mathbf{r},t)}{\partial t}. \quad \text{(A14)}$$

Use equation (A9) to obtain

$$T(\mathbf{r},t) = \frac{\partial}{\partial t}\int_0^t \phi(\mathbf{r},t-\gamma,\gamma)\,d\gamma \quad \text{(A15)}$$

which is Duhamel's theorem. If differentiation is performed under the integral sign, equation (A15) becomes

$$T(\mathbf{r},t) = F(\mathbf{r}) + \int_0^t \frac{\partial}{\partial t}\phi(\mathbf{r},t-\gamma,\gamma)\,d\gamma. \quad \text{(A16)}$$

If only one of the boundary conditions, say, at the surface $i = 1$, is non-homogeneous and a function of time only (i.e. $f_1(t)$), then equation (A16) can be written as

$$T(\mathbf{r},t) = F(\mathbf{r}) + \int_0^t f_1(\gamma)\frac{\partial}{\partial t}\phi(\mathbf{r},t-\gamma)\,d\gamma \quad \text{(A17)}$$

where $\phi(\mathbf{r},t)$ is the solution of the auxiliary problem (A2) with $f_i(\mathbf{r},\gamma)$ replaced by the Dirac delta function δ_{1i}. Clearly, equation (A17) is the specific form of Duhamel's theorem used in equation (5a) in the text.

EFFETS NON-FOURIER SUR LA TEMPERATURE VARIABLE RESULTANT D'UN FLUX PERIODIQUE TOUT-OU-RIEN

Résumé—Les températures variables résultant d'une condition aux limites de flux tout-ou-rien se rencontrent dans plusieurs applications, et parmi d'autres, le frittage des catalyseurs et l'utilisation des impulsions laser pour traiter les semi-conducteurs. Dans de telles conditions, la durée des impulsions est si courte (picoseconde–nanoseconde) que le phénomène classique de diffusion thermique disparait et la nature ondulatoire de la propagation d'énergie caractérisée par l'équation hyperbolique de la chaleur gouverne la distribution de la température dans le milieu. On présente ici une solution analytique explicite pour un problème de conduction thermique linéaire et variable dans un milieu semi-infini soumis à un flux périodique tout-ou-rien à la frontière $x = 0$, en résolvant l'équation hyperbolique de la chaleur. Le cas non-linéaire pour un rayonnement de la surface est étudié numériquement.

EINFLUSS EINES PERIODISCHEN EIN-AUS-SCHALTENS DES WÄRMESTROMS AUF DEN ZEITLICHEN TEMPERATURVERLAUF IN EINEM KÖRPER

Zusammenfassung—Zeitliche Temperaturverläufe, welche von einem periodischen Ein- und Aus-Schalten eines Wärmestroms als Randbedingung herrühren, sind vielfach anzutreffen: unter anderem beim Sintern von Katalysatoren, was häufig beim Ausbrennen von Koks vorkommt und beim Einsatz von Laserimpulsen zum Brennen von Halbleitern. In solchen Situationen ist die Impulsdauer so klein (z.B. Pico-, Nanosekunden), daß das klassische Wärmeausbreitungsphänomen zusammenbricht und die Wellennatur der Energieausbreitung nach der hyperbolischen Wärmeleitgleichung die Temperaturverteilung in dem Medium bestimmt. In dieser Arbeit wird für ein lineares transientes Wärmeleitproblem in einem halbunendlichen Medium, das einem periodischen Ein- und Aus-Schalten des Wärmestroms an der Grenzfläche $x = 0$ unterworfen ist, eine explizite analytische Lösung der hyperbolischen Wärmeleitgleichung vorgestellt. Der nichtlineare Fall, der den zusätzlichen Einfluß der Oberflächenstrahlung an eine externe Umgebung berücksichtigt, wird numerisch untersucht.

ВЛИЯНИЕ ЯВЛЕНИЙ, НЕ ПОДЧИНЯЮЩИХСЯ ЗАКОНУ ФУРЬЕ, НА НЕСТАЦИОНАРНОЕ ТЕМПЕРАТУРНОЕ ПОЛЕ, ВЫЗВАННОЕ ПЕРИОДИЧЕСКИМ ТЕПЛОВЫМ ПОТОКОМ

Аннотация—Нестационарное температурное поле, возникающее при наличии периодического теплового потока на границе, широко применяется на практике, например, при спекании катализатора, при выгорании кокса, а также при использовании лазерных импульсов для отжига полупроводников. В тех случаях, когда длительность импульсов настолько мала (т.е. от пикосекунд до наносекунд), что классический закон диффузии тепла нарушается, распределение температуры в среде определяется волновой природой распространения энергии и описывается гиперболическим уравнением теплопроводности. В данной работе на основе гиперболического уравнения приводится явное аналитическое решение линейной нестационарной задачи теплопроводности в полубесконечной среде, находящейся под воздействием периодического потока тепла на границе $x = 0$. Нелинейный случай, рассматривающий наличие излучения с поверхности в окружающую среду, изучается численно.